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Testing for phase synchronization

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Abstract

Phase synchronization analysis is frequently applied to data originating for example from Physics and Life Sciences. Statistical properties for quantities measuring phase synchronization have not been revealed. We derive an analytic significance level for a frequently used phase synchronization measure. Its performance is demonstrated for a system of coupled oscillators.

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1. Introduction

The field of Nonlinear Dynamics has brought to the forefront novel concepts, ideas, and techniques to analyze and characterize time series of complex dynamic systems. Especially synchronization analysis to detect interactions between nonlinear self-sustained oscillators has made its way into the daily routine in many investigations [1–3].

Following the observations and pioneering work of Huygens, the process of synchronization has been found in many different systems such as systems exhibiting a limit cycle or a chaotic attractor. Several different types of synchronization have been found for these systems ranging from phase synchronization as the weakest form of synchronization via lag synchronization to generalized or complete synchrony [4–9].

Thereby, phase synchronization analysis has gained particular interest since it relies only on very weak coupling between the oscillators. It has been shown that even chaotic oscillators are able to synchronize their phases for considerably weak coupling between them [4]. To quantify the process of synchronization, different measures have been proposed [10,11]. Two frequently used measures are a measure based on entropy and a measure based on circular statistics, which is the so called mean phase coherence [12]. Both measures quantify the sharpness of peaks in distributions of the phase differences. In the following we concentrate on the mean phase coherence.

The mean phase coherence is normalized to $[0, 1]$ with a value of one indicating high synchrony. Hardly any work is devoted to the statistical properties of the mean phase coherence. However, a proper statistical assessment of the results obtained by phase synchronization analysis is an indispensable prerequisite for a reliable application to empirical data. Values of zero or one are hardly observable. Contamination with noise, either observation noise or dynamic noise, alone prevents the clear-cut decision about the presence of phase synchronization.

Several approaches are conceivable that might be able to infer phase synchronization. Frequently used approaches based on surrogate data might give a hint whether or not an observed value for the mean phase coherence is significantly different from zero [11,13,14]. However, there are pitfalls and limitations related to surrogate data tests that have been dealt with by [15,16]. Surrogate techniques like phase randomization suffer from the problem, that not only the interaction between the systems is neutralized but also the systems themselves are lin-

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earized. This might lead to a rejection of the null-hypothesis of independent nonlinear systems which is investigated in this Letter.

A second approach to test for phase synchronization is based on testing for peaks in the distribution of the phase differences. This can be achieved by comparing the phase difference distribution with a uniform distribution. The Kolmogorov–Smirnov test [17] is utilized in this study. Third, a test suggested by Refs. [18,19] based on the asymptotic distribution of the mean phase coherence is also investigated.

For the latter two tests the test statistics is derived under the assumption of independent samples. Thus, they are not expected to work properly in the case of dynamical systems. We propose a theoretical approach utilizing the asymptotic properties of the estimator of the mean phase coherence. These theoretical considerations lead directly to a derivation of a critical value for a given α-significance level that can be calculated in a numerically efficient way. It allows detection of phase synchronization in noisy systems based on non-repetitive realizations of the systems. Similarities to the test statistics presented in Refs. [18,19] are discussed.

We illustrate the performance of this significance level in an application to stochastic synchronizing Rössler oscillators. For these noisy oscillators the onset of phase synchronization is determined utilizing this significance level.

The Letter is organized as follows. After a brief introduction of the mean phase coherence in Section 2, we discuss alternative naïve tests that fail, i.e., they produce false positive conclusions, in Section 3. Thus, motivating the necessity for a novel test, we derive the asymptotic distribution under the hypothesis of absent phase synchronization for the mean phase coherence in Section 4. Due to the central limit theorem on functional spaces, the rigorous derivation of this distribution is possible for a wide range of data generating processes. The process dependent parameters that emerge during the derivation can be estimated from data in a numerical efficient way. This issue is addressed in Section 5, followed by the application of the proposed significance level to a coupled Rössler system in Section 6.

2. Phase synchronization: Mean phase coherence

In order to detect phase synchronization between two coupled self-sustained oscillatory systems a suitable definition of phase and amplitude of a real-valued observed signal is required. This can be realized, if the considered oscillations are characterized by a narrow frequency band [20,21]. Let $x(t)$ be the real-valued signal satisfying the mentioned property. The analytic signal is then given by

$$\psi(t) = x(t) + i\hat{x}(t) = A(t)e^{i\varphi(t)},$$

where $A(t)$ is the amplitude and $\varphi(t)$ the phase. The imaginary part of the analytic signal can be obtained by the Hilbert transform [22]

$$\hat{x}(s) = \pi^{-1}\text{P.V.}\int \frac{x(t)}{s-t}dt$$

of the signal, in which P.V. refers to Cauchy's principle value. The phase $\varphi(t)$ now yields a suitable basis for the synchronization analysis. Please note that the derivations below do not rely on the definition of the phase using the Hilbert transformation. Several other definitions are also conceivable.

Phase synchronization of two coupled, oscillatory systems occurs if the $n:m$ phase locking condition is satisfied [4]

$$\left|n\varphi_x(t) - m\varphi_y(t)\right| = |\Phi_{n,m}| < \text{const}, \tag{1}$$

where $\varphi_x(t)$ and $\varphi_y(t)$ denote the phases of the time series $x(t)$ and $y(t)$, respectively, and n, m are suitable integers. Since the phase is defined between $[-\pi;\pi]$ and in order to correct for phase jumps, induced by the presence of dynamical or observation noise, not the phase difference $\Phi_{n,m}$ itself but

$$\Psi_{n,m} = \Phi_{n,m} \mod 2\pi \tag{2}$$

is investigated. A sharp peak in the distribution of $\Psi_{n,m}$ can be associated with a synchronized state between the oscillators. Here, a commonly used quantity, measuring the sharpness of the distribution of $\Psi_{n,m}$ is the mean phase coherence [12]

$$R_{n,m}^2 = E\left[\cos(\Phi_{n,m})\right]^2 + E\left[\sin(\Phi_{n,m})\right]^2, \tag{3}$$

where $E[\cdot]$ denotes the expectation value. The mean phase coherence is $R_{n,m} = 1$ for a constant phase difference between the two processes and $R_{n,m} = 0$ for a uniformly distributed phase difference in case of non-synchronized oscillators. It has been shown that this quantity is considerably different from zero even in the case of weak coupling, which occurs in the case of phase synchronization.

Let ϕ_i, $i = 1, \ldots, N$, be equidistantly sampled data of $\Phi_{n,m}$, where the time span between the observations is Δt. For sake of simplicity we suppress the subscript n, m in the following. Assuming that the process generating the sample ϕ_i is ergodic, an estimate of R^2 in Eq. (3) is given by

$$\begin{aligned}\hat{R}_N^2 &= \left(N^{-1}\sum_{i=1}^{N}\cos(\phi_i)\right)^2 + \left(N^{-1}\sum_{i=1}^{N}\sin(\phi_i)\right)^2 \\ &= N^{-2}\sum_{i,j=1}^{N}\left(\cos(\phi_i)\cos(\phi_j) + \sin(\phi_i)\sin(\phi_j)\right) \\ &= N^{-2}\sum_{i,j=1}^{N}\cos(\phi_i - \phi_j).\end{aligned} \tag{4}$$

In order to test against the null hypothesis of the absence of phase synchronization, $H_0 : R^2 = 0$, several ad hoc tests have been suggested that are assessed in Section 3. Motivated by the fact that theses tests do not work properly, the asymptotic distribution of the estimate $\hat{R}_N^2$ under H_0 is derived in Section 4.

3. Motivation and naïve tests

Often used tests for the presence or absence of phase synchronization are based on surrogate data, e.g., [11,14]. These tests suffer from the problem, that they are based on too strict mathematical assumptions in contrast to [15,16]. The inference from surrogate data tests are, thus, possibly hazardous [23]. Here, we concentrate on the most naïve approach of surrogates, the Fourier transformation based surrogates. We tested the null hypothesis using 100 surrogates in the case of absent coupling between two stochastic Rössler oscillators [24]

$$\begin{aligned}\dot{x}_{1,2} &= -\omega_{1,2} y_{1,2} - z_{1,2} + \sigma \eta_{1,2},\\ \dot{y}_{1,2} &= \omega_{1,2} x_{1,2} + a y_{1,2},\\ \dot{z}_{1,2} &= b + (x_{1,2} - c) z_{1,2}.\end{aligned} \tag{5}$$

Dynamic noise influence is modeled by Gaussian distributed random variables $\eta_{1,2} \sim \mathcal{N}(0,1)$ leading to the variance σ^2 of the noise term $\sigma\eta_{1,2}$. Using $a = 0.15$, $b = 0.2$, $c = 10$, and $\omega_{1,2} = 1 \pm 0.015$ Hz leads to chaotic oscillations for the deterministic system [4]. The sampling rate is 10 Hz. For the synchronization analysis only the x-components of the Rössler oscillators are examined. Without loss of generality, the phases are estimated using the Hilbert transformation [22].

We investigated sample sizes between 100 and 1,000,000 data points with a standard deviation of the noise of $\sigma = 0.6$. The fraction of rejections of the null hypothesis was 100% for all simulations. Thus, using the standard version of a surrogate data test falsely rejects the null hypothesis in all investigated settings and is inappropriate for synchronization analysis, since it tests for linearity of the involved processes.

The Kolmogorov–Smirnov test [17] for comparison of the phase difference distribution to a uniform distribution on $[-\pi, \pi]$ also leads to 100% rejections of the null hypothesis. Again, we analyzed two uncoupled stochastic Rössler systems and tested sample sizes between 100 and 1,000,000 data points with a standard deviation of the noise of 0.6. The same result was obtained for the naïve test suggested in Refs. [18,19] based on the distribution

$$2N\hat{R}_N^2 \overset{d}{\approx} \chi_2^2, \tag{6}$$

where χ_2^2 denotes the χ^2-distribution with two degrees of freedom. The latter two tests assume independence of the phase difference values which is not the case for dynamical systems. Our simulation study demonstrates that also a large sample size does not provide a way out of this dilemma.

Thus, we derive an analytic significance level in the following that takes explicitly account for the dependence structure of the phase differences for dynamical processes.

4. The distribution of $\hat{R}_N^2$ under H_0

Formulating the phase-difference ϕ_i at time $t_i = i\Delta t$ as an increment process $\phi_i = \phi_{i-1} + \Delta\phi_i$, than under H_0 the increments $\Delta\phi_i$ are strictly stationary for all sampling intervals Δt. Now, consider ϕ_i as a stochastic process and let us assume that the increments $\Delta\phi_i$ are representing an α-mixing process [25]. Precisely, let $\mathcal{F}_l^m = \sigma(\phi_l, \ldots, \phi_m)$ denote the smallest sigma-algebra such that all random variables $\phi_l, \ldots, \phi_m$ are measurable for some $0 \leqslant l \leqslant m$. The process ϕ_i is said to be α-mixing if the mixing coefficient

$$\alpha(k) = \sup_{l \geqslant 0} \sup \left\{ P(A \cap B) - P(A)P(B)\colon\ A \in \mathcal{F}_0^l,\ B \in \mathcal{F}_{l+k}^\infty \right\}$$

satisfies $\lim_{k\to\infty} \alpha(k) = 0$, where $P(\cdot)$ denotes the probability measure. In other words, the statistical dependencies are vanishing for infinitely distant events. Under this condition, it is possible to derive the asymptotic distribution of R_N^2 in the absence of phase synchronization.

As proven in Appendix A we can replace the generally unknown evolution of ϕ by the following drift diffusion process

$$d\tilde{\phi}_t = \omega\, dt + \sqrt{D}\, dW_t, \tag{7}$$

where ω is the mean angular velocity of the phase difference, dW_t is the increment of the Brownian motion and D the diffusion constant. The phases ϕ_i can be approximated by $\phi_i \approx \tilde{\phi}_{i\Delta t} = \tilde{\phi}_t$ leading to the asymptotic distribution of $\hat{R}_N^2$ under the null hypothesis H_0. The procedure of estimating the coefficients ω and D from empirical data is addressed below. Additionally, the initial distribution of $\phi_0 = 0$ with probability one can be assumed without loss of generality, since an over-all phase cancels out calculating the mean phase coherence.

To determine the distribution $\hat{R}_N^2$ under H_0 consider the following random variables

$$\begin{aligned} X_N &= N^{-1} \sum_{i=1}^{N} \cos(\phi_i) \quad \text{and}\\ Y_N &= N^{-1} \sum_{i=1}^{N} \sin(\phi_i). \end{aligned} \tag{8}$$

The solution of Eq. (7) is given by $\phi_i \sim \mathcal{N}(\omega t_i, D t_i)$, where $\mathcal{N}(\mu, \sigma^2)$ denotes the Gaussian distribution with mean μ and variance σ^2. Thus, for the phase model under H_0, Eq. (7), the expectation values of $E[X_N]$ and $E[Y_N]$ yield

$$\begin{aligned} E[X_N] &= N^{-1} \sum_{j=1}^{N} \cos(\omega t_j) e^{-\frac{D}{2} t_j} \quad \text{and}\\ E[Y_N] &= N^{-1} \sum_{j=1}^{N} \sin(\omega t_j) e^{-\frac{D}{2} t_j}, \end{aligned} \tag{9}$$

where $t_j = j\Delta t$. The latter expressions are due to

$$\frac{1}{\sqrt{2\pi}\sigma} \int \cos(x) e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \cos(\mu) e^{-\frac{\sigma^2}{2}}$$

and

$$\frac{1}{\sqrt{2\pi}\sigma} \int \sin(x) e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \sin(\mu) e^{-\frac{\sigma^2}{2}}.$$

Expressing $\cos(\omega t_j)$ and $\sin(\omega t_j)$ by their polar representations and evaluating the geometric sums $\sum_{j=1}^{N} (\cdot)^j$ in Eq. (9) leads to

$$\begin{aligned} E[X_N] &= \frac{1}{N} \left(\frac{f(t_1) - f(t_N) + e^{-D\Delta t}(f(t_{N-1}) - 1)}{(1 - e^{-D\Delta t/2})^2} \right)\\ &= \mathcal{O}(N^{-1}) \end{aligned}$$

and

$$E[Y_N] = \frac{1}{N}\left(\frac{g(t_1) - g(t_N) + e^{-D\Delta t} g(t_{N-1})}{(1 - e^{-D\Delta t/2})^2}\right) = \mathcal{O}(N^{-1})$$

with $f(t_j) = \cos(\omega t_j) e^{-\frac{D}{2} t_j}$ and $g(t_j) = \sin(\omega t_j) e^{-\frac{D}{2} t_j}$. Especially the fact that $E[X_N] = \mathcal{O}(N^{-1})$ and $E[Y_N] = \mathcal{O}(N^{-1})$ is important for the covariance matrix that is discussed next.

The covariance matrix of $Z_N = (X_N, Y_N)$ can be calculated using $\mathbf{C} = \mathrm{Cov}(Z_N) = E[Z_N' Z_N] - E[Z_N'] E[Z_N]$, where Z_N' indicates the transposition of Z_N. Since $E[Z_N] = \mathcal{O}(N^{-1})$ it can be approximated by its second moment with a remainder of order N^{-2}. Thus, $\mathbf{C} = \mathrm{Cov}(Z_N) = E[Z_N' Z_N] + \mathcal{O}(N^{-2})$ and we obtain

$$\mathbf{C} = N^{-2} \sum_{i,j=1}^{N} \begin{pmatrix} E[\cos(\phi_i)\cos(\phi_j)] & E[\cos(\phi_i)\sin(\phi_j)] \\ E[\cos(\phi_i)\sin(\phi_j)] & E[\sin(\phi_i)\sin(\phi_j)] \end{pmatrix} + \mathcal{O}(N^{-2}). \tag{10}$$

Expanding the diagonal entries of the covariance matrix, the expectation values can be represented by $\frac{1}{2}(E[\cos(\phi_i - \phi_j)] \pm E[\cos(\phi_i + \phi_j)])$. The sum in Eq. (10) contains the case where $i = j$, for which follows that $\mathbf{C} = \mathcal{O}(N^{-1})$ and, thus, $N\mathbf{C} = \mathcal{O}(1)$. Thereby, it is guaranteed that Eq. (10) can be approximated by neglecting the term of $\mathcal{O}(N^{-2})$.

Since $\cos(\phi_i)$ and $\sin(\phi_i)$ are strongly mixing sequences, the central limit theorem for mixing processes [29] holds and, therefore, Z_N converges in distribution to the bivariate normal distribution

$$\sqrt{N} Z_N \xrightarrow{d} \mathcal{N}(0, N\mathbf{C}) \quad \text{as } N \to \infty. \tag{11}$$

Since $\mathbf{C}$ is positive definite and symmetric, we can decompose $\mathbf{C} = \mathbf{QDQ}'$, where $\mathbf{Q}$ is orthogonal and $\mathbf{D}$ is diagonal. Setting

$$\tilde{Z}_N \mathbf{D}^{\frac{1}{2}} \mathbf{Q}' = Z_N \tag{12}$$

it follows that $\tilde{Z}_N = (\tilde{X}_N, \tilde{Y}_N) \sim \mathcal{N}(0, \mathbf{1})$ for $N \to \infty$. This is due to the fact that the positive definiteness guarantees that the inverse $(\mathbf{D}^{\frac{1}{2}}\mathbf{Q}')^{-1}$ exists. Hence, $\tilde{Z}_N = (\mathbf{D}^{\frac{1}{2}}\mathbf{Q}')^{-1} Z_N$ and by Eq. (11), $E[\tilde{Z}_N] \to 0$ for $N \to \infty$. Moreover, $E[\tilde{Z}_N' \tilde{Z}_N] = E[Z_N' (\mathbf{QDQ}')^{-1} Z_N] = E[Z_N' (\mathbf{C})^{-1} Z_N] \to \mathbf{1}$ for $N \to \infty$. Since the first and the second moment coincides with the moments of the standard Gaussian distribution and the space of Gaussian distributed random variables is closed with respect to linear transformations, we can state from Eq. (11) that $Z_N \xrightarrow{d} \mathcal{N}(0, \mathbf{1})$. According to Eq. (4) and by the definition of Z_N we can represent the estimator of the mean phase coherence by $\hat{R}_N^2 = Z_N Z_N'$. Inserting Eq. (12) in the previous expression, we obtain $\hat{R}_N^2 = \tilde{Z}\mathbf{D}\tilde{Z}'$. The mean phase coherence $\hat{R}_N^2$ can therefore be quantified using the eigenvalues

$$\hat{R}_N^2 = \lambda_1 \tilde{X}^2 + \lambda_2 \tilde{Y}^2,$$

where λ_1, λ_2 are the eigenvalues of $\mathbf{C}$, given by

$$\lambda_{1/2} = \frac{\mathrm{tr}\,\mathbf{C}}{2} \pm \sqrt{\frac{(\mathrm{tr}\,\mathbf{C})^2}{4} - \det \mathbf{C}}. \tag{13}$$

The eigenvalues λ_1, λ_2 determine the asymptotic distribution of the mean phase coherence under H_0. The distribution of $\hat{R}_N^2$ can be approximated by a superposition of two χ^2-distributions with one degree of freedom, χ_1^2. Since $\lambda_{1/2} > 0$, we can further estimate an upper limit of this distribution by

$$\hat{R}_N^2 \overset{d}{\approx} \mathrm{tr}\,\mathbf{C} \cdot \chi_1^2. \tag{14}$$

Note, that for independent realizations with $\lambda_1 = \lambda_2$ the distribution yields

$$\hat{R}_N^2 \overset{d}{\approx} \frac{\chi_2^2}{2N}, \tag{15}$$

a statistics that was suggested by [18,19].

To obtain the asymptotic distribution for non-independent realizations the trace of the covariance matrix given by Eq. (10)

$$\mathrm{tr}\,\mathbf{C} = N^{-2} \sum_{i,j=1}^{N} \left(E\left[\cos(\phi_i)\cos(\phi_j)\right] + E\left[\sin(\phi_i)\sin(\phi_j)\right]\right) = N^{-2} \sum_{i,j=1}^{N} E\left[\cos(\phi_i - \phi_j)\right] \tag{16}$$

has to be calculated to estimate the asymptotic properties of $\hat{R}_N^2$ under H_0. The approximation in Eq. (14) combined with the relation (16) yields

$$\hat{R}_N^2 \overset{d}{\approx} R_N^2 \cdot \chi_1^2. \tag{17}$$

Note that this expression reveals the exact mean of $\hat{R}_N^2$, thus Eq. (14) also yields the exact expectation value, since $E[\chi_1^2] = 1$ under the null hypothesis. For the phase diffusion the distribution of the phase difference above is normally distributed with mean $\omega|t_i - t_j|$ and variance $D|t_i - t_j|$, compare Eqs. (8) and (9). We therefore obtain,

$$\mathrm{tr}\,\mathbf{C} = N^{-2} \sum_{i,j=1}^{N} e^{-\frac{D}{2}|t_i - t_j|} \cos\left(\omega|t_i - t_j|\right) = \frac{1}{N} + \frac{2}{N} \sum_{s=1}^{N-1} \left(1 - \frac{s}{N}\right) e^{-\frac{D}{2} t_s} \cos(\omega t_s).$$

Abbreviating $\xi = e^{-\frac{D}{2}\Delta t + i\omega\Delta t}$ and $f(t_j) = e^{-\frac{D}{2} t_j} \cos(\omega t_j)$,

$$\mathrm{tr}\,\mathbf{C} = N^{-1}\left(\frac{1}{2} + \xi \frac{1 - \xi^{N-1}}{1 - \xi} - \xi \frac{1 - \xi^{N+1}}{N(1 - \xi)^2} + \frac{\xi^N}{1 - \xi}\right) + \text{c.c.} = N^{-1}\left(1 + 2\frac{f(t_1) + f(t_N) - e^{-D\Delta t}}{(1 - e^{-D\Delta t/2})^2}\right) + \mathcal{O}(N^{-2})$$

can be calculated finally, where c.c. is referred to as the complex conjugation of the previous expression. This distribution of the estimated mean phase coherence can be approximated by

$$\hat{R}_N^2 \sim N^{-1}\left(1 + 2\frac{f(t_1) + f(t_N) - e^{-D\Delta t}}{(1 - e^{-D\Delta t/2})^2}\right)\chi_1^2. \tag{18}$$

This approximation is valid only for a large sample size N.

To obtain a sufficient approximation of the distribution of $\hat{R}_N^2$ under H_0, the mean angular velocity ω and the diffusion constant D have to be reliably estimated.

5. The estimation of ω and D

The estimation of ω can be performed by identification of the linear trend ωt_i for $i = 1, \dots, N$ in ϕ_i obtained by linear regression, such that

$$\hat{\omega} = \frac{\sum_{i=1}^{N} t_i \phi_i}{\sum_{i=1}^{N} t_i^2}, \tag{19}$$

where again $t_i = i\,\Delta t$. Since linear regression is used, the variance of the estimator scales with $1/N$. Stationarity was used here, whereas for the estimation of the diffusion constant D the functional central limit theorem has to be taken into account. Since D is related to the variance of the phase increments $\Delta\phi_i = \phi_i - \phi_{i-1}$,

$$\begin{aligned} D &= \lim_{N\to\infty} \frac{1}{\Delta t N} \operatorname{Var}\left(\sum_{i=1}^{N} \Delta\phi_i\right) \\ &= \lim_{N\to\infty} \frac{1}{\Delta t} \sum_{k=-N+1}^{N-1} \left(1 - \frac{k}{N}\right)\gamma(k) \\ &= \frac{1}{\Delta t} \sum_{k=-\infty}^{\infty} \gamma(k), \end{aligned} \tag{20}$$

where $\gamma(k) = E[(\Delta\phi_i - E[\Delta\phi_i])(\Delta\phi_{i+k} - E[\Delta\phi_{i+k}])]$ is the auto-covariance function of the phase increments $\Delta\phi_i$. The auto-covariance function is a substantial part of Eq. (20), such that the correlations of the phase increments cannot be neglected in the estimation.

To deal with the problem of correlated phase increments, non-overlapping blocks are build up out of the phase increments in a manner such that we achieve approximately independent blocks. This approach is similar to the one used in block bootstrap [31,32]. Finding such non-overlapping blocks is possible if the particular time series is strongly mixing which was one of the central requirements of the functional central limit theorem. It is further assumed without loss of generality that for a given block-length l the number of blocks b is an integer number, otherwise the time series of the increments can sufficiently be truncated. We define the total phase increment of each block by

$$\delta_j = \sum_{i=1}^{l} \Delta\phi_{(j-1)l+i}, \quad j = 1, \dots, b = \frac{N}{l}. \tag{21}$$

The empirical variance of δ_j divided by $l\Delta t$ therefore yields an appropriate estimate for D and is given by

$$\hat{D} = \frac{1}{l\Delta t} b^{-1} \sum_{j=1}^{b} (\delta_j - l\hat{\omega}\Delta t)^2. \tag{22}$$

Here, the free parameter, the block-length l, has to be selected. If, e.g., the block-length was too small, the estimate of D could be strongly biased due to the correlations. On the other hand, if l is too large $\hat{D}$ itself shows a rather high variance. The optimal block-length should balance both effects. This can be achieved if the mean-squared-error MSE = Variance + Bias^2 is minimized with respect to the block-length l. In Appendix B, an approximation of MSE is derived which yields

$$\begin{aligned} \mathrm{MSE} &\approx (\Delta t)^{-2}\left(l^{-2}C_1 + \frac{2l}{N}C_2\right), \\ C_1 &= \left(\sum_{k=-\infty}^{\infty} |k|\gamma(k)\right)^2, \quad \text{and} \\ C_2 &= \left(\sum_{k=-\infty}^{\infty} \gamma(k)\right)^2. \end{aligned} \tag{23}$$

The optimal block-length is given by the minimum of Eq. (23), thus $l_{opt} = (NC_1/C_2)^{\frac{1}{3}}$. Certainly, both constants C_1 and C_2 are unknown in the first place, moreover if C_2 was known the diffusion constant could be calculated using Eq. (20) directly. Instead, rough estimates of these constants are used to determine an almost optimal block-length, where the outcome of Eq. (20) is directly linked to the estimate of C_2. Under the assumption that the auto-covariance function decays exponentially, such a rough estimate is given by the following scheme [32]:

(1) Estimate the auto-correlation function of the increments $\Delta\phi_i$.
(2) Fit $\varphi(k) = \varphi^k$ to the envelope of the auto-correlation function.
(3) Compute the estimate of the optimal block-length $\hat{l}$ by

$$\hat{l} = (4N)^{1/3}\left(\frac{\varphi}{1-\varphi} + \frac{\varphi^2}{(1-\varphi)^2}\right)^{2/3} \times \left(1 + 2\frac{\varphi}{1-\varphi}\right)^{-2/3}.$$

The variance of the estimator of D can be derived from the results in Appendix B. The variance scales with $1/b$, the number of independent blocks. The MSE of D scales with $N^{-2/3}$.

In summary, the statistical properties of the mean phase coherence have been derived. Please note that we do not cover all possible situations that might be faced in real-world applications. For instance, noise induced phenomena are not addressed in the above derivation. However, as long as the assumptions of the test statistics are fulfilled, a reasonable significance level will be obtained. To this aim, the parameters that are defined by the realization of the process, i.e., the mean angular velocity and the diffusion coefficient, can be estimated following the procedures above. An application to empirical data is feasible. To illustrate the performance of the proposed significance level, we test it on simulated data in the following section.

6. Performance of the significance level

To assess the proposed critical value at a particular significance level, a system of two coupled stochastic Rössler oscillators [24]

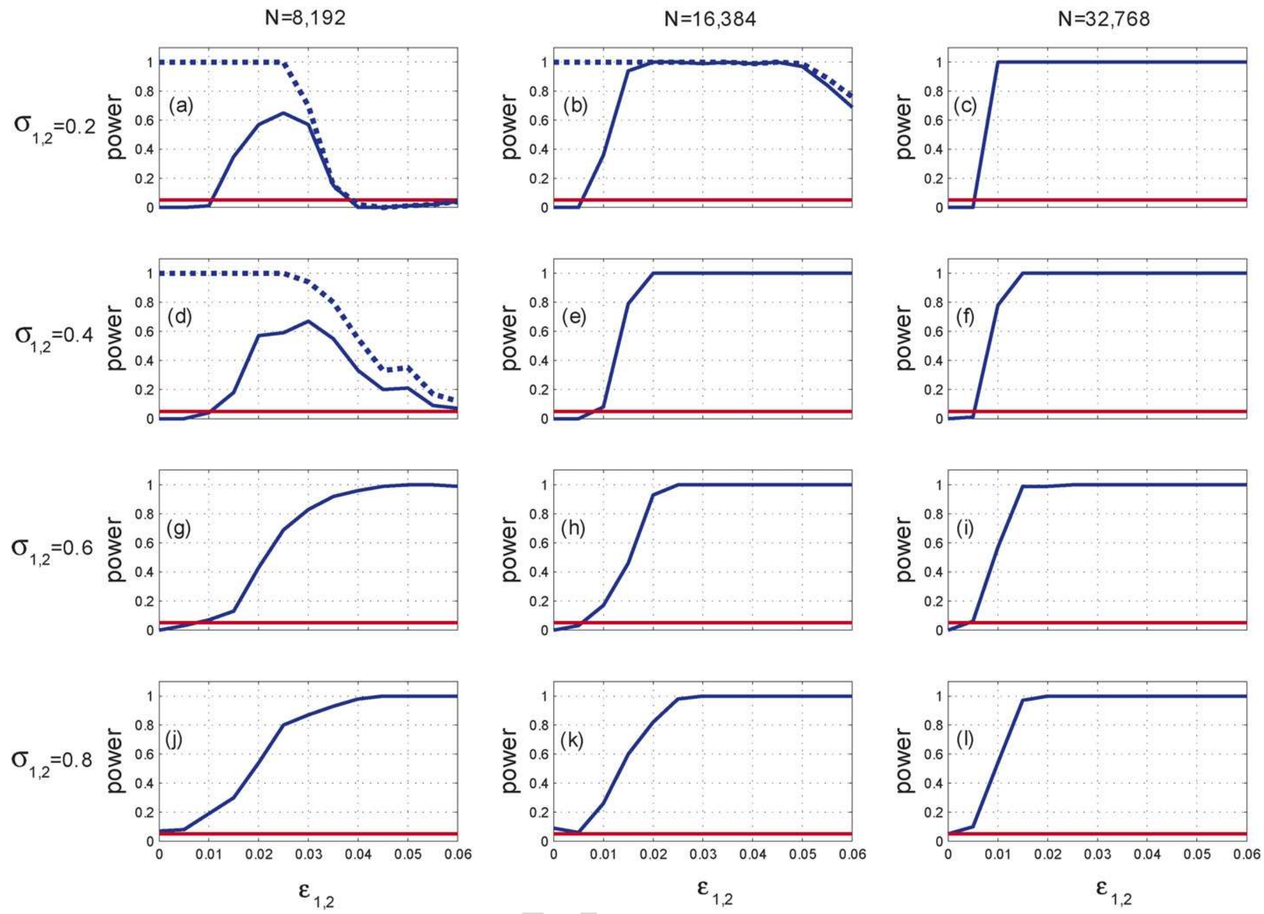

Fig. 1. Power of the proposed significance level in dependence on data length and noise level. In subsequent rows the noise strength is varied, while between columns the sample size is different. The horizontal line indicates a 5% level, which is the fixed significance level in our study. The continuous line shows the power of the proposed significance level dependent on the coupling strength. The dotted lines in (a), (b), and (d) indicate the fraction of critical values below one.

$$\begin{aligned}\dot{x}_{1,2} &= -\omega_{1,2}y_{1,2} - z_{1,2} + \varepsilon_{1,2}(x_{2,1} - x_{1,2}) + \sigma_{1,2}\eta_{1,2},\\ \dot{y}_{1,2} &= \omega_{1,2}x_{1,2} + ay_{1,2},\\ \dot{z}_{1,2} &= b + (x_{1,2} - c)z_{1,2}\end{aligned} \tag{24}$$

is investigated. Dynamic noise influence is modeled by Gaussian distributed random variables $\eta_{1,2} \sim \mathcal{N}(0,1)$ leading to the variance $\sigma_{1,2}^2$ of the noise term $\sigma_{1,2}\eta_{1,2}$. Using $a = 0.15$, $b = 0.2$, $c = 10$, and $\omega_{1,2} = 1 \pm 0.015$ Hz leads to chaotic oscillations for the deterministic system [4]. The sampling rate is 10 Hz. For the synchronization analysis only the x-components of the Rössler oscillators are examined. Without loss of generality, the phases are estimated using the Hilbert transformation [22].

The coupling strength between the two oscillators is modeled by the parameters $\varepsilon_{1,2}$. For this system the onset of phase synchrony is achieved for a bidirectional coupling strength of 0.03 in the noise-free case. We varied the bidirectional coupling strength between both Rössler oscillators, the noise of each Rössler oscillator $\sigma_{1,2}\eta_{1,2}$ as well as the length of the sample simulated for each Rössler oscillator to quantify the coverage as well as the power of the proposed significance level. The coverage measures the number of false positive conclusions in the absence of phase synchronization and has to be controlled, i.e. there should be not more than α% false positive conclusions for an α-significance level. The α-significance level can be derived from Eq. (14) by

$$R^{\alpha}_{N,\mathrm{crit}} = \operatorname{tr}\mathbf{C} \cdot \chi^2_{1,\alpha}, \tag{25}$$

where $\chi^2_{1,\alpha}$ denotes the α-quantile of the χ^2-distribution with one degree of freedom. The power measures the ability to detect a true phase synchronization between the oscillators by the fraction of correctly rejected null-hypotheses. In the following, 100 realizations for every parameter set were simulated to determine power and coverage of the proposed significance level.

In Fig. 1 the results are shown. The sampling length is varied between columns ranging from 8192 to 32,768. Subsequent rows reflect different noise levels $\sigma_{1,2}$ in the range between 0.2 and 0.8. The bidirectional coupling strength is varied for each parameter combination in the range from 0 to 0.06. These coupling strengths are sufficient to warrant phase synchrony in the noise free case. The horizontal line in each subplot corresponds to the 5% level of significance derived in the previous sections.

First, we would like to emphasize that for absence of coupling between the Rössler oscillators the coverage keeps to or below the 5%-significance level. In other words, the critical value for the mean phase coherence prevents erroneous conclusions in the case of absent coupling. Second, for high coupling strength and sufficiently large noise and sampling interval the power reaches values of 100%, which clearly indicates that a high power of the proposed significance level is achieved for reasonable parameter combinations (Fig. 1(c, e, f, g–l)). The steepness of several power curves especially for a large sample size, moreover, emphasizes the performance for the level of significance in general, see for instance Fig. 1(c).

A more detailed characterization of the power curves yields some interesting properties. To simplify our statements, we focus our interpretations on non-zero coupling strength between the oscillators in the following if not otherwise stated. In general, the performance increases for increasing amount of data and decreases for higher noise levels. This behavior is intuitively expected and for example revealed in the difference between Fig. 1(e) and (k). For constant sampling size, the noise level is twice as high in Fig. 1(k). The coupling value for which a 100% power is reached is shifted from the coupling strength of 0.02 (e) to the slightly higher coupling strength 0.03 (k). However, the onset of detection of an interaction between the oscillators is shifted to slightly lower coupling strengths for the higher noise variance. This effect is due to finite size effects which distinguishes the behavior of the significance level when it has not reached its asymptotic regime. The finite size effects are mainly influenced by the diffusion constant. If the diffusion constant is large, the amount of data is effectively higher. There are more independent data points since the system is faster mixing. For the Rössler system this aspect has been discussed in Ref. [33].

The consequence that the assumptions are not fulfilled is even more illustrative if small sample sizes are investigated. Higher noise levels can even increase the performance not only for the onset of detection of interaction but also for higher coupling strengths (1st column). Note that the power in (a) decays after an increase up to 0.65 at a coupling strength of 0.025 again when the coupling strength is further increased. This is caused by critical values of or higher than one, preventing detection of significant results. This effect emerges in the simulations shown in Fig. 1(a), (b), and (d). The dotted line shows the fraction of critical values that are below one. In those cases, where the power decreases again, the critical value is, thus, to a certain extent close to or higher than one. The equation for the critical values enables understanding of this phenomenon. The χ^2-distribution multiplied by the trace of matrix $\mathbf{C}$ is not limited to one as the mean phase coherence. This occurs especially for low noise and small sample sizes and represents a finite size effect.

For the investigated ranges of couplings an increase in the noise, however, allows the statistics to become applicable and therefore shifts the power curve to higher values of the power for higher coupling strengths. Once a particular noise value is exceeded, in our case 0.6, the power does not decrease any more. Alternatively the asymptotic behavior can also be achieved, when the number of data points is enlarged. This is revealed by the plots in the first row. Doubling the sample size is almost sufficient to shift this drop in power to very high coupling strength. Four times larger sample size guarantees the power to stay at 100% for the investigated coupling strength.

For absent coupling between the oscillators the coverage keeps below the 5%-significance level for small noise variance. This is due to the approximation of the sum of two independent χ_1^2-distributions with one χ_1^2-distribution. The approximation is conservative and thus, the significance level is conservative in those cases. Since the power increases rapidly this conservative behavior for absent coupling does not hamper the applicability of the proposed significance level. Therefore, it is not necessary to approximate the sum of two independent χ_1^2-distributions more exactly, which would lead to a much more complex expression for the significance level. Moreover, for higher noise variances that are usually expected in the majority of applications the coverage is 5% as desired by the significance level.

For higher coupling strengths than shown in Fig. 1 the statistics is not necessarily in its asymptotic. A critical value of or higher than one would be observed preventing any conclusions about a coupling between the oscillators. Whenever a critical value of or higher than one is obtained, further investigations are necessary as to which extent this belongs to a true result or to a missing of asymptotic behavior of the system. In this sense the significance level prevents false positive conclusions as it indicates when its assumptions are not fulfilled. Moreover, the length of the necessary segments to estimate the diffusion coefficient is an indicator whether or not the process is sufficiently mixing. If the length of the segments is too large, one should be cautious when drawing conclusions.

Additionally, the proposed significance level is not capable of distinguishing coupling between oscillators and for instance a signal propagation. One has to ensure in the first place, that one is in the regime of coupled synchronizing oscillators. This problem is very common in time series research and there are suggestions how to distinguish synchronizing oscillators and, e.g., signal propagation in the first place [34].

7. Conclusion

We derived the distribution of the test statistics for the mean phase coherence that leads to a critical value for a corresponding significance level that allows to test for a non-zero synchronization value. Its performance has been demonstrated in a simulation study based on coupled stochastic Rössler oscillators. The coverage of the significance level is conservative. Moreover, the level of significance is characterized by a steep increase in power. One major advantage of the proposed significance level lies in the fact that the suggested procedure provides information about its applicability to the problem at hand. If the segment length necessary for the estimation of the diffusion term is too large compared to the time series length, indicating that the system is either non-mixing or that the mixing rate is too slow, the proposed significance level should not be used which is indicated by the proposed procedure. To put it the other way around, if the diffusion constant is large, short data segments are sufficient to reliably apply the proposed significance level. In contrast to the naïve tests, false positive conclusions about the synchronization in cases, where these conclusions cannot be inferred, are prevented by the proposed test when it is indicating that it was not applicable.

In summary, the proposed significance level works well for a large variety of coupling strengths, noise variances, and sample sizes. It provides, thus, a powerful test for the presence of phase synchrony.

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Appendix A. Functional limit theorem

Assume that the process space is endowed with the Skorohod topology. The Skorohod topology is defined by the metric $d(\cdot,\cdot)$ on the space $\mathbb{D}[0,1]$ of cadlag functions on $[0,1]$. A cadlag function on $[0,1]$ is a real-valued function that fulfills

- $\lim_{s\uparrow s_0} x(s)$ exists for every $s_0 \in (0,1]$,
- $\lim_{s\downarrow s_0} x(t) = x(s_0)$ exists for every $s_0 \in (0,1]$.

In general, we have to rescale the time t, i.e., $s = t/t_{\max} \in [0,1]$. Let Λ denote the class of strictly increasing continuous mappings of $[0,1]$ onto itself [26,27]. Then, the distance $d(x,y)$ is the infimum of those positive ε for which there exists an $\lambda \in \Lambda$ with

$$\sup_{s\in[0,1]} \bigl\{|\lambda(s) - s|\bigr\} \leqslant \varepsilon \quad \text{and}$$
$$\sup_{s\in[0,1]} \bigl\{\bigl|x(s) - y\bigl(\lambda(s)\bigr)\bigr|\bigr\} \leqslant \varepsilon$$

for $x(s), y(s) \in \mathbb{D}[0,1]$.

Then, the functional central limit theorem [27,28] states that the sufficiently rescaled sum of the centered increments converge weakly to Brownian motion on $[0,1]$, i.e., there exist functions $\lambda_n \in \Lambda$ such that

$$\lim_{n\to\infty} x_n\bigl(\lambda_n(s)\bigr) = x(s) \quad \text{and}$$
$$\lim_{n\to\infty} \lambda_n(s) = s$$

uniformly in s. The x_n are the sequence that converge weakly to Brownian motion.

Appendix B. Derivation of the mean squared error

In this appendix, the mean-squared-error for the estimation of the diffusion constant

$$\begin{aligned}
\hat{D} &= \frac{1}{l\Delta t} b^{-1} \sum_{j=1}^{b} (\delta_j - l\hat{\omega}\Delta t)^2 \\
&= \frac{1}{l\Delta t} b^{-1} \sum_{j=1}^{b} \sum_{r,m=1}^{l} (\Delta\phi_{(j-1)l+r} - \hat{\omega}\Delta t) \\
&\quad \times (\Delta\phi_{(j-1)l+m} - \hat{\omega}\Delta t) \\
&= \frac{1}{\Delta t} b^{-1} \sum_{j=1}^{b} \sum_{k=-l+1}^{l-1} \hat{\gamma}_j(k)
\end{aligned}$$

is derived, where $\hat{\gamma}_j(k)$ is the empirical auto-covariance function of block j. The bias $E[\hat{D}] - D$ can be calculated using the value of D as derived in Eq. (20). The bias then reads

$$\begin{aligned}
E[\hat{D}] - D &= \frac{1}{\Delta t} b^{-1} \sum_{j=1}^{b} \sum_{k=-l+1}^{l-1} E\bigl[\hat{\gamma}_j(k)\bigr] - D \\
&= \frac{1}{\Delta t} b^{-1} \sum_{j=1}^{b} \sum_{k=-l+1}^{l-1} \biggl(1 - \frac{|k|}{l}\biggr)\gamma(k) - D \\
&\approx \frac{1}{\Delta t} \sum_{k=-\infty}^{\infty} \biggl(1 - \frac{|k|}{l}\biggr)\gamma(k) - D \\
&= -\frac{1}{l\Delta t} \sum_{k=-\infty}^{\infty} |k|\gamma(k).
\end{aligned}$$

The variance can also be derived analytically. It becomes

$$\begin{aligned}
\mathrm{Var}(\hat{D}) &= \biggl(\frac{1}{\Delta t}\biggr)^2 b^{-2}\, \mathrm{Var}\Biggl(\sum_{j=1}^{b} \sum_{k=-l+1}^{l-1} \hat{\gamma}_j(k)\Biggr) \\
&\approx \biggl(\frac{1}{\Delta t}\biggr)^2 b^{-1}\, \mathrm{Var}\Biggl(\sum_{k=-\infty}^{\infty} \hat{\gamma}_j(k)\Biggr) \\
&= \biggl(\frac{1}{\Delta t}\biggr)^2 b^{-1}\, \mathrm{Var}\bigl(\mathrm{Per}(0)\bigr),
\end{aligned}$$

where the latter expression denotes the variance of the periodogram at frequency zero. This is asymptotically given by

$$\mathrm{Var}\bigl(\mathrm{Per}(0)\bigr) = 2\Biggl(\sum_{k=-\infty}^{\infty} \gamma(k)\Biggr)^2,$$

see, e.g., [30]. Setting C_1 and C_2 as in Eq. (23) and substituting $b = N/l$, an approximation of the mean-squared-error is finally given by

$$\begin{aligned}
\mathrm{MSE} &= \mathrm{Variance} + \mathrm{Bias}^2 \\
&\approx (\Delta t)^{-2}\biggl(l^{-2}C_1 + \frac{2l}{N}C_2\biggr).
\end{aligned}$$

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